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Nonlinear Stability of Parallel Flows by High-Order Amplitude Expansions

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The Landau-Stuart theory and its subsequent modifications suffer from some restrictions and from the nonuniqueness in determining higher-order terms of the amplitude expansions, which limit the range of applicability as well as the validity of the results. In the present paper, a well-defined amplitude is introduced a priori. In this way, uniqueness for terms of any order is achieved. Moreover, Watson's method is no longer restricted to almost neutral disturbances. This offers not only more accurate approximations and numerical studies on convergence but, as a consequence, a whole series of new applications. As a first example, the nonlinear equilibrium states of the plane Poiseuille flow are investigated. The numerical results are discussed in context with the author's solutions of the nonlinear equations and with special emphasis on the convergence of Landau's series.

I. Introduction

THE ultimate objective of the nonlinear stability theory is to bridge the gap between the predictions of the linear theory and the observed process of transition to turbulence. Progress, however, is rather modest and, although transition is strongly three-dimensional and breakdown is highly localized, the nonlinear theory for parallel flows is as yet primarily concerned with the amplitude-dependence of the stability with respect to a single two-dimensional harmonic disturbance.

One of the most helpful, and most usual, tools for the treatment of this problem are expansions in the small amplitude of the disturbance. The use of such expansions started with a heuristic consideration by Landau,[1] which led to the first terms of the perturbation series

$$\frac{\mathrm{d}A}{\mathrm{d}t} = A\left(a_0 + a_1 A^2 + \ldots\right) \tag{1}$$

for the real amplitude $A(t)$. The derivation of Landau's Eq. (1) from the Navier-Stokes equations and, most important, a procedure for determining the constant a_1 was presented by Stuart.[2] Whereas Stuart has taken the small amplification rate as a perturbation parameter, a very similar formulation for the infinite series (1) by directly expanding in the small disturbance amplitude was given by Watson.[3]

The first applications provided fundamental results on the supercritical stability of the circular Couette flow[4] and on the subcritical instability of the plane Poiseuille flow.[5,6] Stewartson[7] regards the Stuart-Watson theory as "the most important theoretical advance since the discovery of the Tollmien-Schlichting waves" and reports on various other applications, modifications, and extensions of the method.

There are, however, some unintelligible properties of the current amplitude expansion methods, as well as objections against recent applications, which are mostly concerned with 1) the restriction to almost neutral disturbances, 2) the convergence of the formal expansion series, 3) the validity of the results, and 4) the nonuniqueness of higher-order terms. As suggested by Reynolds and Potter,[5] restriction 1 can be replaced by the alternative 1') restriction to the equilibrium solutions ($\mathrm{d}A/\mathrm{d}t=0$) of Eq. (1), but neither of these formulations covers the analysis of the growth of small disturbances definitely away from the neutral curve. Even more restricted are applications to interacting wave pairs or wave packets.

Joseph and Sattinger[8] use another perturbation method for constructing the equilibrium solutions of Eq. (1), but their proofs can be modified to show the convergence of the perturbation series suggested by Reynolds and Potter for sufficiently small amplitudes. Although the arguments can possibly be extended to the more general case $\mathrm{d}A/\mathrm{d}t \neq 0$, the radius of convergence remains unknown for the time being. Moreover, there are not yet any error estimates assuring validity (point 3) of the results obtained from truncated series.

In principle, some light on question 3 could be shed by a comparison of different approximations. Anyway, this requires overcoming considerable difficulties in the numerical evaluation of a sequence of (homogeneous and inhomogeneous) Orr-Sommerfeld equations. But moreover, there arises a problem of nonuniqueness (point 4) in determining higher-order terms which restricts the application of the amplitude expansion method to the first nonlinear approximation. Only Eagles[9] has given reasonable arguments for a determination of the second Landau constant a_2 in Eq. (1) for the Taylor-Couette flow which have neither been extended to higher-order terms nor to the more complicated problem of parallel flows.

It is shown herein that essential shortcomings of the amplitude expansion methods can be removed by introducing a priori a well-defined amplitude. It is not only that terms of any order are now uniquely determined in both Watson's method and the Reynolds-Potter approach, but, in addition, the former method is no longer restricted to almost neutral disturbances. The basic idea of the modified formulation is, as in earlier work, discussed for the plane Poiseuille flow, but can simply be adapted to other parallel flows and cases like the Taylor-Couette flow as well. The paper does not give the full equations or detailed derivations and its first part should be considered as complementary to the papers by Watson[3] and Reynolds and Potter.[5]

The plane Poiseuille flow is also chosen as a suitable example for the numerical studies, since the nonlinear stability of this flow has been investigated successfully both experimentally[10] and theoretically.[11,12] An efficient and

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accurate spectral-collocation method[12] is used for calculating terms up to the 15th order in the amplitude in order to gain some insight into the convergence properties of the perturbation series.

II. Linear Stability Problem

We consider the flow of an incompressible fluid of kinematic viscosity ν in a plane channel of half-width h. The flow is driven by a fixed mean pressure gradient. The basic flow is parabolic with the maximum velocity $\bar{U}$ at the channel centerline. Nondimensionalization is based on h and $\bar{U}$ and the Reynolds number is defined by $R = \bar{U}h/\nu$. The two-dimensional fluid motion can be specified by a stream function $\psi(x,y,t)$, where x is the streamwise direction, y is the distance from the centerline vertical to the walls at $y = \pm 1$, and t is the time.

The linear stability theory adds small disturbances ψ' to the stream function $\Psi(y)$ of the basic flow,

$$\psi = \Psi(y) + \psi'(x,y,t) \tag{2}$$

where the disturbances are in the form of Tollmien-Schlichting waves

$$\psi' = A(t)\phi_{10}(y)e^{i(\alpha x - \omega_0 t)} \tag{3}$$

of wave number α and frequency ω_0. The amplitude $A(t)$ varies exponentially,

$$A(t) = A_0 e^{a_0 t} \tag{4}$$

and satisfies the first term of Landau's series, Eq. (1). At given R and α, the complex eigenvalue $\lambda_0 = a_0 - i\omega_0$, as well as the related eigenfunction ϕ_{10}, result from solving the Orr-Sommerfeld equation

$$L(\alpha)\phi_{10} = \left\{ \frac{1}{R}\left(\frac{d^2}{dy^2} - \alpha^2\right)^2 - \left(i\alpha\frac{d\Psi}{dy} + \lambda_0\right)\left(\frac{d^2}{dy^2} - \alpha^2\right) + i\alpha\frac{d^3\Psi}{dy^3}\right\}\phi_{10} = 0 \tag{5}$$

with homogeneous boundary conditions $\phi_{10} = d\phi_{10}/dy = 0$ at $y = \pm 1$. Unique determination of ϕ_{10} requires some normalization, e.g.,

$$\phi_{10}(y_0) = 1 \tag{6}$$

at a suitable position $y = y_0$. We observe, that for a given ψ' there is a direct interrelation between modulus and argument of the normalization, Eq. (6), and amplitude and phase of the wave, Eq. (3), respectively.

For the plane Poiseuille flow, it is well known that a symmetric eigenfunction is related to the principal eigenvalue. Therefore, the channel center $y_0 = 0$ is a suitable location for normalization (6). The regions of stability ($a_0 < 0$) and instability ($a_0 > 0$) in an α,R-plane are separated by a neutral curve $a_0(\alpha,R) = 0$, which shows a critical Reynolds number $R_{c0} = 5772$ at $\alpha_{c0} = 1.02$.

III. Nonlinear Problem

For a growing disturbance, $a_0 > 0$, the exponential growth of the amplitude, Eq. (4), will not indefinitely continue due to the action of nonlinear terms. It is expected that the flow finally attains a stable equilibrium state of amplitude A^* (Fig. 1a). The existence of equilibrium states in a close neighborhood of the neutral curve is guaranteed by statements of the mathematical bifurcation theory.[8] An observable example of a stable secondary flow is the Taylor-vortex state which develops from circular Couette flow. A similar situation may occur in the stable subcritical domain where $a_0 < 0$. There, the nonlinear terms may act destabilizing (Fig. 1b) and the amplitude A^* of the unstable secondary flow is a threshold deciding on the growth or decay of a finite disturbance.

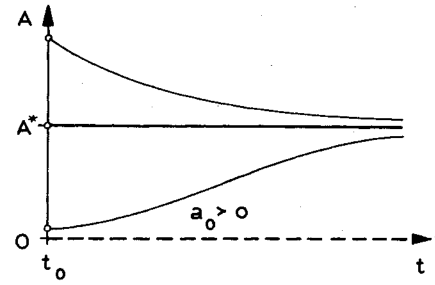

Fig. 1a Amplitude A vs time of disturbances attracted by a stable equilibrium state at A^*.

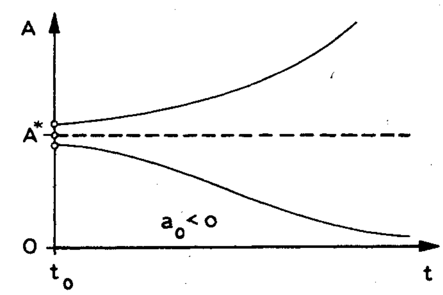

Fig. 1b Amplitude A vs time of disturbances repelled from an unstable equilibrium state at A^*.

The secondary flow is an alternative solution of the Navier-Stokes equations, which is periodic in x and t and can be regarded as steady in a Galilean frame moving with the unknown phase velocity $c^* = \omega^*/\alpha$. Its amplitude $A^* = A^*(\alpha,R)$ maps out a neutral surface in the A,α,R-space, which has the same fundamental significance for the nonlinear stability problem as the neutral curve has for the linear problem.

The calculation of the secondary flow is based on the Fourier series

$$\psi(x,y,t) = \sum_{n=-\infty}^{\infty} \psi_n e^{in\alpha\xi} \quad \xi = x - \frac{\omega}{\alpha}t \tag{7}$$

where $\psi_{-n} = \tilde{\psi}_n$ must be satisfied in order to obtain a real solution and the tilde denotes the complex conjugate. According to Fig. 1a, one may consider the unsteady problem

$$\psi_n = \psi_n(y,t) \quad \omega = \omega(t) \quad t \to \infty \tag{8}$$

and follow the growth in time of an initially small disturbance. This approach is, however, conceptually less appropriate for calculating the repellent threshold states (Fig. 1b). Stable as well as unstable cases may be covered by a direct attack on the steady problem

$$\psi_n = \psi_n^*(y) \quad \omega = \omega^* \tag{9}$$

Approximate solutions can be found by a Fourier truncation method, i.e., by solving a coupled system of $N+1$ nonlinear differential equations of the Orr-Sommerfeld type for ω^*, ψ_n^*, $n \le N$. Numerical results were reported by Grohne[13] for $N = 1$, by Zahn et al.[14] for $N \le 2$, and by the author[11,12] for $N \le 4$. Without going into the details of these results, it should be noticed that the Fourier series seems to converge quite rapidly at small amplitudes. But even with sophisticated numerical techniques, this method exhausts the capacity of today's computers at rather low approximation levels.

IV. Perturbation Method

An alternative way to calculate the Fourier coefficients are perturbation methods, such as Watson's method for the unsteady problem. After substituting $A(t)$ for t and exploiting the quadratic nonlinearity of the Navier-Stokes equations, one obtains

$$\psi_n(y,A) = A^n \phi_n(y,A) \tag{10}$$

and the Linstedt-Poincaré method leads to the formal expansions

$$\phi_n = \sum_{m=0}^{\infty} \phi_{nm}(y) A^{2m} \tag{11}$$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = A \sum_{m=0}^{\infty} a_m A^{2m} \tag{12}$$

$$\frac{\mathrm{d}(\omega t)}{\mathrm{d}t} = A \sum_{m=0}^{\infty} \omega_m A^{2m} \tag{13}$$

The functions $\phi_{nm}(y)$ are governed by a set of linear differential equations which can be solved in a certain sequence. A uniformly valid expansion of the solution can be obtained only with specific values of the constants a_m, ω_m which, at the same time, determine the behavior of amplitude and frequency.

In a standard procedure, one would calculate the terms of the following scheme line by line from left to right.

		ψ_0	ψ_1	ψ_2	ψ_3	ψ_4	ψ_5	...
1		ϕ_{00}						
A	a_0,ω_0		ϕ_{10}					
A^2		ϕ_{01}		ϕ_{20}				
A^3	a_1,ω_1		ϕ_{11}		ϕ_{30}			
A^4		ϕ_{02}		ϕ_{21}		ϕ_{40}		
A^5	a_2,ω_2		ϕ_{12}		ϕ_{31}		ϕ_{50}	
...				...				

(14)

The first line contains only the basic flow, $\phi_{00} = \Psi$, whereas the second line requires the solution of the linear stability problem. The terms of order $0(A^2)$ can then be calculated in succession. Before proceeding to ϕ_{11}, the constants a_1,ω_1 must be known. Surprisingly, it turns out from Watson's formulation that, in general, these constants cannot be determined. Only in the special case $a_0 = 0$ and, by reasons of steadiness, for

$$m|a_0| \ll 1 \qquad m = 1 \tag{15}$$

these values are obtained from a solvability condition. But even then, the nonlinear distortion ϕ_{11} of the fundamental and all the higher-order terms ϕ_{nm}, $m>0$, are not uniquely determined. It can be seen from Eq. (14) that this implies a Fourier truncation at $N=2$.

The modified formulation of Reynolds and Potter[5] for the steady problem

$$\mathrm{d}A^*/\mathrm{d}t = 0 \tag{16}$$

avoids the restriction, Eq. (15), on nearly neutral disturbances, i.e., the constants a_m,ω_m can be evaluated for any values of R and α. The amplitude of the secondary flow is then obtained from

$$\sum_{m=0}^{\infty} a_m (A^*)^{2m} = 0 \tag{17}$$

The nonuniqueness of the higher-order terms, however, also remains present in this formulation.

This nonuniqueness imposes not only additional restrictions on the expansion methods concerning the smallness of the amplitude and the number of harmonics, but prevents any comparison of different approximations which could be helpful in estimating the validity of the results. For the steady problem, the first nonlinear approximation has been compared[11] with results of the Fourier truncation method including the same number $(N=2)$ of harmonics. The agreement is very poor, except in the immediate neighborhood of the neutral curve, although the Reynolds-Potter method is not restricted to small amplification rates. It has also been shown by energy considerations that meaningless results are obtained throughout the parameter domain off from the neutral curve. At large wave numbers, it turns out that even an extremely small amplitude is no guarantee for a physically relevant solution.

V. Definition of the Amplitude

The formal comparison of the Fourier truncation method with current amplitude expansion methods also discloses the reason for the unexpected nonuniqueness in a well-posed problem. A careful analysis of the formulations shows that the definition of the amplitude has been solely based on the normalization, e.g. Eq. (6), imposed on the eigenfunction of the linear Orr-Sommerfeld problem. Thus, the amplitude has been typically derived from

$$\psi_1(y_0,A) = A(t)\phi_{10}(y_0) + 0(A^3) \tag{18}$$

$$\phi_{10}(y_0) = 1 \tag{19}$$

Obviously, this definition is insufficient. It can easily be seen that the nonlinear contributions to the fundamental cannot be uniquely determined in this way. If the normalization at y_0 is replaced by $\phi_{10}(y_0) = 1 + kA^2$, the amplitude remains unchanged, although the arbitrary constant k is of the same order as $\phi_{11}(y_0)$.

One can, however, measure the size of the fundamental unambiguously by introducing a new definition of the amplitude, e.g.,

$$\psi_1(y_0,A) = A(t)\phi_1(y_0,A) \tag{20}$$

$$\phi_1(y_0,A) = 1 \tag{21}$$

This choice is in no way mandatory, but it is computationally simple, physically evident, and consistent with earlier analyses as far as the determination of a_1,ω_1 is concerned. A comparison of the condition, Eq. (21), with the series, Eq. (11), for ϕ_1 leads in the lowest order to the well-known normalization, Eq. (19), of ϕ_{10} and to an additional infinite set of conditions

$$\phi_{1m}(y_0) = 0 \qquad m>0 \tag{22}$$

The important role of these conditions can be illustrated best by an examination of the equations leading to $\lambda_m = a_m - i\omega_m$ and $\phi_{1m}(y)$. In Watson's method, these equations are of the type

$$L(\alpha)\phi_{1m} - 2ma_0(\phi''_{1m} - \alpha^2\phi_{1m})$$

$$= \lambda_m(\phi''_{10} - \alpha^2\phi_{10}) + f_{1m} \qquad m>0 \tag{23}$$

where $L(\alpha)$ denotes the Orr-Sommerfeld operator, Eq. (5), the prime indicates differentiation with respect to y, and f_{lm} is considered as a known function of y at the stage when this equation is to be solved. The boundary conditions $\phi_{lm}=\phi'_{lm}=0$ at $y=\pm 1$ are identical to those for ϕ_{10}.

Let us first consider some point of the neutral curve where $a_0=0$. Then, Eq. (23) reduces to the inhomogeneous Orr-Sommerfeld equation

$$L(\alpha)\phi_{lm}=\lambda_m(\phi''_{10}-\alpha^2\phi_{10})+f_{lm} \tag{24}$$

Equations of the same type arise in the method of Reynolds and Potter at arbitrary points of the R, α plane. Both methods take profit of the fact that the related homogeneous problem, Eq. (5), is solvable, and, consequently, the constants a_m,ω_m must be chosen so that the well-known orthogonality condition is satisfied. Whereas a_l,ω_l and, hence, the right-hand side of Eq. (24) for $m=1$ are uniquely determined in this way, the solution $\phi_{11}(y)$ can only be found to within an arbitrary constant multiple of the solution $\phi_{10}(y)$ of the homogeneous problem, Eq. (5). Without any further information on ϕ_{11}, this would cause nonuniqueness of all the higher-order terms containing this function. Now, with the additional condition $\phi_{11}(y_0)=0$ according to Eq. (22), this arbitrariness is completely removed since $\phi_{10}(y_0)\neq 0$. By repeating these arguments at the stages $m>1$, it is easy to show that Eq. (22) guarantees uniqueness of terms of arbitrary order in Watson's method with the restriction, Eq. (15), as well as in the Reynolds-Potter approach to equilibrium states, Eq. (16). It should be noticed that physically less transparent conditions related to Eqs. (21, 19, and 22) can be found in the perturbation method suggested by Joseph and Sattinger[8] for constructing equilibrium states, and in the application to the plane Poiseuille flow by Chen and Joseph.[15] Although Chen and Joseph point out various differences between their method and previous work on amplitude expansions, this remarkable fact is not indicated.

Besides the removal of the nonuniqueness, there is another very important consequence of the new conditions Eq. (22) for Watson's approach to the unsteady problem where $a_0\neq 0$. In this case, the solution of Eq. (23) can be combined from the solutions of

$$L(\alpha)\chi_{m0}-2ma_0(\chi''_{m0}-\alpha^2\chi_{m0})=\phi''_{10}-\alpha^2\phi_{10} \tag{25}$$

$$L(\alpha)\chi_{ml}-2ma_0(\chi''_{ml}-\alpha^2\chi_{ml})=f_{lm} \tag{26}$$

subject to the boundary conditions $\chi_{mi}=\chi'_{mi}=0$ at $y=\pm 1$, $i=1,2$. Both these problems are solvable and, in particular, $\chi_{m0}=-\phi_{10}/(2ma_0)$. Then, the solution of Eq. (23) takes the form

$$\phi_{lm}(y)=\chi_{ml}(y)-(\lambda_m/2ma_0)\phi_{10}(y) \tag{27}$$

Watson concluded that outside the small band, Eq. (15), along the neutral curve the constants λ_m can be arbitrarily chosen, e.g., specified to be zero, leaving the nonlinear effects on the amplitude Eq. (12) and the frequency Eq. (13) undetermined. Now, with conditions (22), these constants received definite values

$$\lambda_m=a_m-i\omega_m=2ma_0\chi_{ml}(y_0) \tag{28}$$

which remove any arbitrariness in the functions $\phi_{lm}(y)$. Thus, Watson's method is now applicable for studying the growth of amplified disturbances definitely away from the neutral curve and can be extended for investigations of wave pairs and wave packets as well. The invalidity of the method for decaying disturbances, which was pointed out by Davey and Nguyen[16] can be overcome along the lines indicated by Itoh[17] for disturbances with spatially varying amplitude.

VI. Numerical Results

The amplitude expansion method for the steady, as well as the unsteady, problem can now be applied for any values of R and α and can be formally extended to arbitrary order. However, the numerical treatment requires extreme care. Already the setup of the differential equations and the manipulation of the increasing number of terms should be carried out by a computer program. If the series (12) is truncated at $m\leq M$ and all terms of order $0(A^{2M+1})$ are taken into account, then $(M+1)(M+2)-1$ differential equations (mostly of the Orr-Sommerfeld type) must be solved in succession. In order to prevent a dramatic error propagation, very high accuracy must be achieved by the numerical method, whereas the large number of differential equations requires high efficiency of the method. These requirements are well satisfied by the Chebyshev collocation method[12] successfully applied for treating the nonlinear differential equations obtained by the Fourier truncation method.

The numerical results reported shortly were obtained by single precision calculations (Univac 1108, 27 bit mantissa). The computer program consists of about 250 cyclically arranged Fortran statements. The calculations are based on Reynolds and Potter's formulation for the steady problem, which bears the advantage that the differential operators applied to ϕ_{nm} are independent of m. The discussion concentrates only on the constants a_m and the amplitude A^*, although the complete solution contains other data (ω_m,ω^*, $\phi^*_{nm}(y)$) as well.

The real Landau constants $a_m, m\leq 7$ at the critical point R_{c0},α_{c0} for the plane Poiseuille flow are shown in Table 1. The result $a_l=29.69$ is in the expected agreement with other results given in the literature.[5,15] The constants, $a_m, m>1$ contradict strictly the general belief that the higher-order terms are of minor importance. The values increase faster than R^m, as can be seen from the values $b_m=a_m/R^m$ in Table 1. It is recommended to rescale the problem by introducing

$$\epsilon=RA^{*2} \tag{29}$$

and to replace the truncated series (17) by

$$\sum_{m=0}^{M} b_m\epsilon^m=0 \tag{30}$$

It is obvious that reasonable results can be obtained only for sufficiently small values of ϵ.

For the infinite series, $M\to\infty$, one would now proceed in three steps:

1) Find the radius ρ of convergence of the formal series

$$\sum_{m=0}^{\infty} b_m r^m$$

2) Find the real solutions ϵ_k of

$$\sum_{m=0}^{\infty} b_m\epsilon^m=0$$

Table 1 Coefficients a_m of Landau's series, modified coefficients b_m, and ratios q_m at the critical point $R=5772.22$, $\alpha=1.02056$

m	a_m	b_m	q_m
0	0	0	—
1	$2.969.10^{1}$	0.005144	—
2	$-2.611.10^{5}$	-0.007836	0.6565
3	$2.424.10^{9}$	0.01261	0.6216
4	$-2.851.10^{13}$	-0.02568	0.4909
5	$3.912.10^{17}$	0.06105	0.4207
6	$-5.758.10^{21}$	-0.1557	0.3921
7	$8.695.10^{25}$	0.4072	0.3823

3) Select the physically meaningful solutions which satisfy $0 \leq \epsilon_k < \rho$.

Obviously, the radius of convergence cannot be strictly concluded from numerical data. The techniques described by Van Dyke[18] can be applied to some extent, but an estimation of the nearest singularity from Domb-Sykes plots requires even higher approximations which should be calculated with double precision. One can, however, conclude that reasonable results are obtained only if the terms $|b_m r^m|$ decay rapidly with increasing m. Root test and ratio test lead to very similar answers, since the ratios $q_m = |b_{m-1}/b_m|$, $m>1$, are relatively close to each other (see Table 1). For numerical convergence, the ratio test requires

$$r < \delta q_M, \quad 0<\delta<1 \quad (M>1) \tag{31}$$

with a sufficiently small value of δ.

Step 2 needs some care, since the truncated version, Eq. (30), may well provide solutions which disappear for $M \to \infty$. These spurious solutions are not necessarily removed by step 3 (see, e.g., the truncated series for e^{-x}). In the present problem, it is known that a unique physical solution bifurcates from the neutral curve. The numerical investigation of various approximations $1 \leq M \leq 7$, in an extended parameter domain provided, in general, either none or only one solution ϵ_0 of Eq. (30) which satisfies $0 \leq \epsilon_0 < q_M$. At the critical point, the physical solution is $\epsilon_0 = 0$, whereas the spurious solutions $\epsilon_1 = 0.657$, 0.571, 0.503 for $M=2$, 4, 6, respectively, are removed by the ratio test.

Table 2 shows the relevant data for the subcritical point $R=5000$, $\alpha=1.12$ which is subject to detailed investigations in the experiments of Nishioka et al.[10] as well as in the author's comparison[11] with the theory. Except for the smaller values of q_m, there seem to be no notable changes in comparison with Table 1. The physical solutions are given in Table 3 for odd M. As a remarkable fact, the approximations with even M provide only pairs of complex conjugate roots. Table 3 shows a rather moderate convergence of the expansion method. Although terms of 15th order are included with $M=7$, the approximation has not yet reached the final value $\epsilon = 0.2370$ obtained by the Fourier truncation method ($N=4$). In this case, the amplitude of the secondary flow is $A^* = 0.0069$, which corresponds to a maximum rms velocity fluctuation of $u' = 0.0216$ in the x direction. The experimental result is $u' = 0.015$ and agrees, accidentally, much better with the value $u' = 0.017$ obtained by the expansion method with $M=1$.

Table 2 Modified coefficients b_m and ratios q_m at the subcritical point $R=5000$, $\alpha=1.12$

m	b_m	q_m
0	−0.002783	—
1	0.01850	—
2	−0.04543	0.4071
3	0.1196	0.3797
4	−0.3514	0.3404
5	1.110	0.3166
6	−3.675	0.3021
7	12.55	0.2929

Table 3 Physical solution ϵ_0 and the maximum rms velocity u' at $R=5000$, $\alpha=1.12$

M	ϵ_0	u'
1	0.1505	0.0173
3	0.1961	0.0197
5	0.2119	0.0204
7	0.2195	0.0208

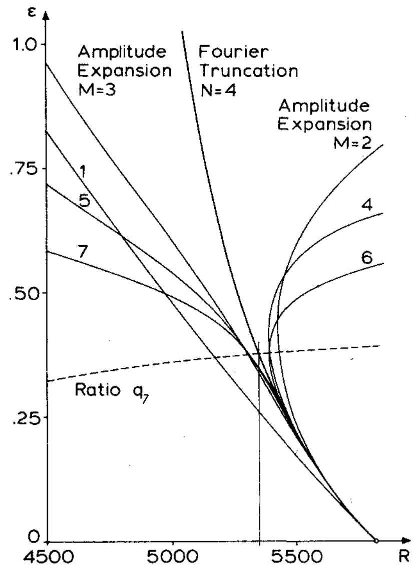

Fig. 2 Solutions $\epsilon_k > 0$ vs Reynolds number R for wavenumber $\alpha = 1$ obtained from different methods and approximations.

The comparison between the results of the Fourier truncation method and various truncations of the expansion series is shown in Fig. 2 for $\alpha = 1$. The convergence of the perturbation method is obvious within a certain distance from the neutral curve. In this range, a considerable improvement of the results can be achieved by including higher-order terms. The convergence domain can be well estimated by the ratio test. Outside of this domain, the approximations seem to converge to a nonphysical final state, which is governed by the large highest-order constants.

Although this comparison shows that the Fourier truncation method is superior to the amplitude expansion method, the improved perturbation method will hold an important place in the analysis of nonlinear stability problems. It is expected that the convergence can be accelerated and the validity of the results checked by more sophisticated criteria. Possibly, the convergence domain can be extended in order to cover the physically relevant range of amplitudes. Most important, however, is the fact that the improved expansion method provides a more rational basis for future applications to single harmonic disturbances, as well as for generalizations toward studies of the nonlinear interaction and coupling of different modes.

References

[1] Landau, L., "On the Problem of Turbulence," *Comptes Rendus de l'Académie des Sciences de l'URSS*, Vol. 44, 1944, pp. 311-314.

[2] Stuart, J. T., "On the Non-Linear Mechanics of Wave Disturbances in Stable and Unstable Parallel Flows," *Journal of Fluid Mechanics*, Vol. 9, Nov. 1960, pp. 353-370.

[3] Watson, J., "On the Non-Linear Mechanics of Wave Disturbances in Stable and Unstable Parallel Flows," *Journal of Fluid Mechanics*, Vol. 9, Nov. 1960, pp. 371-389.

[4] Davey, A., "The Growth of Taylor Vortices in Flow Between Rotating Cylinders," *Journal of Fluid Mechanics*, Vol. 14, Nov. 1962, pp. 336-368.

[5] Reynolds, W. C. and Potter, M. C., "Finite-Amplitude Instability of Parallel Shear Flow," *Journal of Fluid Mechanics*, Vol. 27, Feb. 1967, pp. 465-492.

[6] Pekeris, C. L. and Shkoller, B., "Stability of Plane Poiseuille Flow to Periodic Disturbances of Finite Amplitude in the Vicinity of the Neutral Curve," *Journal of Fluid Mechanics*, Vol. 29, July 1967, pp. 31-38.

[7] Stewartson, K., "Some Aspects of Nonlinear Stability Theory," *Fluid Dynamics Transactions, Polish Academy of Science*, Vol. 7, 1975, pp. 101-128.

[8] Joseph, D. D. and Sattinger, D. H., "Bifurcating Time Periodic Solutions and their Stability," *Archive for Rational Mechanics and Analysis*, Vol. 45, 1972, pp. 79-109.

[9]Eagles, P. M., "On the Stability of Taylor Vortices by Fifth-Order Amplitude Expansions," *Journal of Fluid Mechanics*, Vol. 49, Oct. 1972, pp. 529-550.

[10]Nishioka, M., Iida, S., and Ichikawa, Y., "An Experimental Investigation of the Stability of Plane Poiseuille Flow," *Journal of Fluid Mechanics*, Vol. 72, Dec. 1975, pp. 731-751.

[11]Herbert, Th., "Finite Amplitude Stability of Plane Parallel Flows," *Proceedings of the AGARD Symposium on Laminar-Turbulent Transition*, Paper No. 3, AGARD-CP-224, 1977.

[12]Herbert, Th., "Die neutrale Fläche der ebenen Poiseuille-Strömung," Habilitationsschrift, Universität Stuttgart, 1978.

[13]Grohne, D., "Die Stabilität der ebenen Kanalströmung gegenüber dreidimensionalen Störungen von endlicher Amplitude," AVA Göttingen, Bericht 69 A 30, 1969.

[14]Zahn, J. P., Toomre, J., Spiegel, E. A., and Gough, D. O., "Nonlinear Cellular Motions in Poiseuille Channel Flow," *Journal of Fluid Mechanics*, Vol. 64, June 1974, pp. 319-345.

[15]Chen, T. S. and Joseph, D. D., "Subcritical Bifurcation of Plane Poiseuille Flow," *Journal of Fluid Mechanics*, Vol. 58, April 1973, pp. 337-351.

[16]Davey, A. and Nguyen, H.P.F., "Finite-Amplitude Stability of Pipe Flow," *Journal of Fluid Mechanics*, Vol. 45, Feb. 1971, pp. 701-720.

[17]Itoh, N., "Spatial Growth of Finite Wave Disturbances in Parallel and Nearly Parallel Flows, Part 1. The Theoretical Analysis and the Numerical Results for Plane Poiseuille Flow," *Transactions of the Japan Society for Aeronautical and Space Sciences*, Vol. 17, 1974, pp. 160-174.

[18]Van Dyke, M., "Analysis and Improvement of Perturbation Series," *Quarterly Journal of Mechanics and Applied Mathematics*, Vol. 27, Pt. 4, 1974, pp. 423-450.